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POISSON STABILITY OF REVERSIBLE SYSTEMS†

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An investigation is presented of the stability in Poisson's sense of reversible systems in which the phase volume is not invariant, a particular example of which is non-holonomic systems. Criteria are proposed for the stability of such systems in Poisson's sense, and the existence of integral invariants is discussed.

1. CONSIDER an autonomous system of differential equations

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}) \tag{1.1}$$

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where $\mathbf{X}(\mathbf{x}) \in C^1(D(\mathbf{x}) \subset R^n)$, the domain $D(\mathbf{x})$ is an invariant set, $\mathbf{x} = (x_1, \ldots, x_n)^T$, and

$$\mathbf{X} = (X_1, \ldots, X_n)^T, \qquad t \in R$$

As we know [1, 2], if v is the volume of a domain $\Delta \subset D$:

$$v = \operatorname{mes}(\Delta) = \int_\Delta dx, \qquad \Delta(t=0) = \Delta_0$$

then Liouville's Theorem states that

$$\frac{dv}{dt} = \int_\Delta \operatorname{div} \mathbf{X}\, dx, \quad dx = dx_1 \ldots dx_n$$

and, in particular, if $\operatorname{div}\mathbf{X} \equiv 0$, then v is an invariant of system (1.1):

$$\int_\Delta dx = \int_{\Delta_0} dx$$

and the methods of ergodic theory are applicable [3]. A similar situation arises if $\operatorname{div}\mathbf{X} \not\equiv 0$ and system (1.1) has the invariant [1]

$$v^* = \operatorname{mes}(\Delta) = \int_\Delta \rho(\mathbf{x})\, dx \tag{1.2}$$

where $\rho(\mathbf{x}) \in C^1(D(\mathbf{x}) \subset R^n)$ is a positive function satisfying Liouville's equation [1]

$$\operatorname{div}(\rho \mathbf{X}) = 0 \tag{1.3}$$

Unfortunately, a solution $\rho(\mathbf{x}) \in C^1(D(\mathbf{x}))$ of Eq. (1.3) does not always exist [4, 5]. It will therefore be interesting to pick out a class of systems with ergodic properties—in particular, stable systems in Poisson's sense (Poisson stable systems) [1, p. 363]—without linking the question to the existence of an integral invariant (1.2).

Lemma 1. Let $\Omega \subseteq D(\mathbf{x})$ be a bounded invariant domain of system (1.1). If

$$\sup_{t \in R} \left| \int_0^t \operatorname{div} \mathbf{X}(\mathbf{x}(\tau))\, d\tau \right| < \infty \tag{1.4}$$

$\forall\, \gamma = \{\mathbf{x}(t) : t \in R\} \subset \Omega$, then almost all trajectories $\gamma \subset \Omega$ are Poisson stable.

Proof. Augment system (1.1) by adding the equation

$$dx_{n+1}/dt = -x_{n+1} \operatorname{div} \mathbf{X}(\mathbf{x}) \tag{1.5}$$

thus expressing the entire system in the form

$$\mathbf{x}^{\cdot *} = \mathbf{X}^*(\mathbf{x}^*) \tag{1.6}$$

where $\mathbf{x}^* = (x_1, \ldots, x_{n+1})^T$, $\mathbf{X}^*(\mathbf{x}^*) = (\mathbf{X}, -x_{n+1}\operatorname{div}\mathbf{X})^T \in D \times R$. Since $\operatorname{div}\mathbf{X}^*(\mathbf{x}^*) \equiv 0$, the phase volume of system (1.6) is an invariant.

System (1.1) appears here as a separate constituent of (1.6). Assuming that its solution

$$\mathbf{x} = \varphi(t, \mathbf{x}_0) \in C^{(1,1)}_{t,\mathbf{x}_0}, \qquad \mathbf{x}_0 \in D \tag{1.7}$$

is known, we integrate Eq (1.5). The result is

$$x_{n+1}(t) = x_{n+1}(0) \exp\left(-\int_0^t (\operatorname{div} \mathbf{X}|_{\mathbf{x} = \varphi(\tau, \mathbf{x}_0)})\, d\tau\right) \tag{1.8}$$

It follows from (1.7) and (1.8) that $x_{n+1} \in C_{\mathbf{x}_0}$ and the solution of (1.6)

$$\mathbf{x}^* = \psi(t, \mathbf{x}_0^*), \quad \mathbf{x}_0^* \in D \times R_{x_{n+1}}, \qquad t \in R$$

is a flow ([6, p. 48]; see also [1, p. 346]).

By assumption the solution $x_{n+1}(t)$ is bounded, $\forall\, \gamma \subset \Omega$, $x_{n+1}(0) < \infty$; hence the invariant set (1.6) is also bounded:

$$\Omega^* = \Omega \times x_{n+1},\ x_{n+1}(0) \in]0,\ b[,\quad 0 < b < \infty$$

The set Ω^* has positive Lebesgue measure in R^{n+1}. Therefore, by Poincaré's Recurrence Theorem [1–3, 7], almost all trajectories $\gamma^* \subset \Omega^*$ are Poisson stable. Hence, using the equality [8, p. 310]

$$\operatorname{mes} \Omega^* = \operatorname{mes}_{R^n}(\Omega) \operatorname{mes}_R(x_{n+1})$$

we conclude that almost all trajectories $\gamma \subset \Omega$ are Poisson stable in the sense of relative Lebesgue measure in Ω. This completes the proof of Lemma 1.

Corollary. The conclusion of Lemma 1 remains true if inequality (1.4) holds only up to a set of trajectories of Lebesgue measure zero.

This lemma, considered as a sufficient condition for Poisson stability, is not constructive. It would therefore be interesting to specify a class of systems for which Poisson stability could be investigated without having an explicit expression for $\operatorname{div} \mathbf{X}[\mathbf{x}(t)]$ valid for solutions of system (1.1). It turns out that the reversible systems constitute such a class.

2. Consider the holonomic reversible system [9, p. 83]

$$\mathbf{x}^{\cdot\cdot} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\cdot}),\quad \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}, -\mathbf{y})$$
$$\mathbf{f}(\mathbf{x}, \mathbf{x}^{\cdot}) \in C^1(D(\mathbf{x}, \mathbf{x}^{\cdot}) \subset R^{2n}) \tag{2.1}$$

Expressing (2.1) in the form of equations of the first order

$$\mathbf{x}^{\cdot} = \mathbf{y} \quad \mathbf{y}^{\cdot} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{2.2}$$

we see that replacing t by $(-t)$ in these equations is equivalent to the substitution $(\mathbf{x}, \mathbf{y})^T \to (\mathbf{x}, -\mathbf{y})$. Thus, as a corollary of reversibility, we obtain: with every bounded (unbounded) solution of system (2.2) for $t \in R^+ = [0, \infty[$ one can associate (by the rule $(\mathbf{x}, \mathbf{y})^T \to (\mathbf{x}, -\mathbf{y})^T$) a bounded (unbounded) solution for $t \in R^- =]-\infty, 0]$.

Theorem 1. Let $\Omega \subseteq D(\mathbf{x}, \mathbf{x}^{\cdot}) \subset R^{2n}$ be a bounded invariant domain of system (2.1). Then almost all trajectories $\gamma \subset \Omega$ are Poisson stable.

Before proving this theorem, we need the following lemma.

Lemma 2. Under the assumptions of Theorem 1, for almost all $(\mathbf{x}, \mathbf{x}^{\cdot})^T \subset \Omega$,

$$\sup_{t \in R^+(R^-)} \left| \int_0^t \sum_{i=1}^n \frac{\partial f_i}{\partial x_i^{\cdot}} d\tau \right| < \infty$$

Proof. Augment system (2.2) by adding the equation

$$z^{\cdot} = -z \operatorname{div} \mathbf{F}(\mathbf{x}, \mathbf{y}) \tag{2.3}$$

and then express the augmented system in the form

$$\mathbf{w}^{\cdot} = \mathbf{W}(\mathbf{w}),\quad \mathbf{w} = (\mathbf{x}, \mathbf{y}, z)^T$$
$$\mathbf{W} = (\mathbf{F}, -z \operatorname{div} \mathbf{F})^T,\quad \mathbf{F} = (\mathbf{y}, \mathbf{f}(\mathbf{x}, \mathbf{y}))^T \tag{2.4}$$

Since $\operatorname{div} \mathbf{W} \equiv 0$, the phase volume is an invariant of this system. Since

$$\operatorname{div} \mathbf{F}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \frac{\partial f_i}{\partial y_i}$$
$$\operatorname{div} \mathbf{F}(\mathbf{x}, \mathbf{y})|_{\mathbf{y} \to (-\mathbf{y})} = -\operatorname{div} \mathbf{F}(\mathbf{x}, \mathbf{y}) \tag{2.5}$$

the substitution $t \to -t$ in Eqs (2.4) is equivalent to $(\mathbf{x}, \mathbf{y}, z) \to (\mathbf{x}, -\mathbf{y}, z)$.

By (2.3),

$$z(t) = z(0)\exp\left(-\int_0^t \operatorname{div}\mathbf{F}\,d\tau\right) \tag{2.6}$$

and, by (2.5), replacing t by $(-t)$ in this equation is equivalent to replacing $\operatorname{div}\mathbf{F}$ by $(-\operatorname{div}\mathbf{F})$.

Since the initial system (2.1) is invariant under the substitution of $-t$ for t, so that it is possible to convolve the equations $\mathbf{x}^\cdot = -\mathbf{y}$, $\mathbf{y}^\cdot = -\mathbf{f}(\mathbf{x}, \mathbf{y})$, which are obtained from (2.2) by again replacing t by $(-t)$ in (2.1), it follows that the transformation $(\mathbf{x}, \mathbf{y})^T \to (\mathbf{x}, -\mathbf{y})^T$ leaves any trajectory γ of system (2.2) invariant. Therefore, if we assume that the component $z(t)$ of the solution $\mathbf{w}$ of system (2.4) is not bounded as $t \to \infty(-\infty)$, it follows from (2.6) that, conversely, as $t \to -\infty(\infty)$ it will tend to zero.

Let ω be the image of Ω in $(\mathbf{x}, \mathbf{y})$-space and $\omega^* \subset \omega$ an invariant subset of trajectories of (2.2), such that

$$\inf_{t \in R^+(R^-)} \int_0^t \sum_{i=1}^n \frac{\partial f_i}{\partial y_i}\,d\tau = -\infty \tag{2.7}$$

Suppose that $\operatorname{mes}(\omega^*) > 0$. It then follows from (2.6) and (2.7) that

$$\lim_{t \to -\infty(+\infty)} z(t) = 0, \quad \forall\,(\mathbf{x}_0, \mathbf{y}_0)^T \in \omega^*$$

irrespective of the value of $z(0) \in]0, b[$, $0 < b < \infty$.

Thus,

$$\lim_{t \to -\infty(+\infty)} \operatorname{mes}(\omega^* \times z) = \lim_{t \to -\infty(+\infty)} \operatorname{mes}_{R^n}(\omega^*)\operatorname{mes}_R(z) = 0$$

contrary to the invariance of the phase volume of system (2.4). Thus, if trajectories of (2.2) satisfying (2.7) exist, their Lebesgue measure must be zero. This proves Lemma 2.

We will now prove Theorem 1.

By Lemma 2,

$$\sup_{t \in R} \left| \int_0^t \operatorname{div}\mathbf{F}(\mathbf{x}(\tau), \mathbf{x}^\cdot(\tau))\,d\tau \right| < \infty$$

for almost any trajectory $\gamma = \{(\mathbf{x}(t), \mathbf{x}^\cdot(t)) : t \in R\} \subset \Omega$. Hence, by the Corollary to Lemma 1, we complete the proof of the theorem.

3. Consider a non-holonomic system

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \mathbf{q}^\cdot} - \frac{\partial L}{\partial \mathbf{q}}\right)\delta\mathbf{q} = 0, \quad \mathbf{q} \in D \subset R^n \times R^l \tag{3.1}$$

$$q^\cdot_{n+i} = \sum_{j=1}^n b_{ij}(\mathbf{q})\,q^\cdot_j, \quad i = 1, 2, \ldots, l \tag{3.2}$$

$$L(\mathbf{q}, \mathbf{q}^\cdot) = T(\mathbf{q}, \mathbf{q}^\cdot) - \Pi(\mathbf{q}) = \tfrac{1}{2}\mathbf{q}^{\cdot T} A(\mathbf{q})\,\mathbf{q}^\cdot - \Pi(\mathbf{q})$$

where $L(\mathbf{q}, \mathbf{q}^\cdot)$, $b_{ij}(\mathbf{q}) \in C^2_{\mathbf{q}}(D)$ and the quadratic form $\mathbf{q}^{\cdot T}A(\mathbf{0})\mathbf{q}^\cdot$ is positive-definite. Let us assume first that the Lagrangian L and the coefficients $b_{ij}(\mathbf{q})$ do not depend on the coordinates $q_{n+1}, \ldots, q_{n+l}$. Then system (3.1), (3.2) reduces to Chaplygin's equations [10, 11]:

$$\frac{d}{dt}\frac{\partial L^*}{\partial q^\cdot_j} - \frac{\partial L^*}{\partial q_j} - \sum_{i=1}^l \sum_{m=1}^n \left(\frac{\partial L}{\partial q^\cdot_{n+i}}\right)^* \beta^i_{jm}\,q^\cdot_m = 0 \tag{3.3}$$

$$\beta^i_{jm} = \partial b_{ij}/\partial q_m - \partial b_{im}/\partial q_j, \quad j, m = 1, 2, \ldots, n$$

where the asterisk indicates that the generalized velocities $q^\cdot_{n+i}$ have been eliminated from L and $\partial L/\partial q_{n+i}$ by using the constraint equations (3.2).

As the quadratic form $\mathbf{q}^{\cdot T}A(\mathbf{0})\mathbf{q}^\cdot$ is positive-definite, it follows that Eqs (3.3) are solvable for the highest-order derivatives in some domain $D^*(\mathbf{q}, \mathbf{q}^\cdot) \subset R^{2n}$, and are reduced to the form

$$\mathbf{q}^{\cdot\cdot} = \boldsymbol{\psi}(\mathbf{q}, \mathbf{q}^{\cdot}) = \boldsymbol{\psi}_1(\mathbf{q}) + \boldsymbol{\psi}_2(\mathbf{q}, \mathbf{q}^{\cdot}) \tag{3.4}$$

where $\psi(\mathbf{q}, \mathbf{q}^{\cdot}) \in C^1(D^*(\mathbf{q}, \mathbf{q}^{\cdot}))$ and the function ψ_2 is quadratic in $\mathbf{q}^{\cdot}$. Since $\psi(\mathbf{q}, \mathbf{p}) = \psi(\mathbf{q}, \mathbf{p})$, we thus obtain a reversible system of the form (2.1).

Theorem 2. Let $\Omega \subseteq D^*(\mathbf{q}, \mathbf{q}^{\cdot}) \subset R^{2n}$ be a bounded invariant domain of system (3.3). Then almost all trajectories $\gamma \subset \Omega$ are Poisson stable.

Now suppose that system (3.1), (3.2) does not contain constraints of the type characteristic for Chaplygin's equations. Then we obtain the more general Voronets equations [12, 13]:

$$\frac{d}{dt}\frac{\partial L^*}{\partial q_j^{\cdot}} = \frac{\partial L^*}{\partial q_j} + \sum_{i=1}^{l} \frac{\partial L^*}{\partial q_{n+i}} b_{ij} + \tag{3.5}$$

$$+ \sum_{i=1}^{l}\sum_{m=1}^{n} \left(\frac{\partial L}{\partial q_{n+i}^{\cdot}} \right)^{*} \beta_{jm}^{i} q_m^{\cdot}, \quad j = 1, 2, \ldots, n$$

$$\beta_{jm}^{i} = \frac{\partial b_{ij}}{\partial q_m} - \frac{\partial b_{im}}{\partial q_j} + \sum_{\mu=1}^{l} \left(\frac{\partial b_{ij}}{\partial q_{n+\mu}} b_{\mu m} - \frac{\partial b_{im}}{\partial q_{n+\mu}} b_{\mu j} \right)$$

These equations must be considered together with the constraint equations (3.2). In this case the equations of motion do not reduce to equations of the form (2.1).

Theorem 3. Let $\Omega \subseteq D^*(\mathbf{q}, \mathbf{q}^{\cdot}) \subset R^{n+l} \times R^n$ be a bounded invariant domain of system (3.2), (3.5). Then almost all trajectories $\gamma \subset \Omega$ are Poisson stable.

Proof. Making the substitution

$$\partial L^* / \partial q_j^{\cdot} = p_j$$

we transform system (3.2), (3.5 to the form

$$q_j^{\cdot} = \frac{\partial H}{\partial p_j}, \quad j = 1, 2, \ldots, n; \quad q_{n+i}^{\cdot} = \sum_{j=1}^{n} b_{ij} \frac{\partial H}{\partial p_j} \tag{3.6}$$

$$p_j^{\cdot} = -\frac{\partial H}{\partial q_j} - \sum_{i=1}^{l} \frac{\partial \Pi}{\partial q_{n+i}} b_{ij} + \varphi_j(\mathbf{q}, \mathbf{p})$$

$$H = \sum_{j=1}^{n} q_j^{\cdot} p_j - L^* = \frac{1}{2} \mathbf{p}^T B(\mathbf{q}) \mathbf{p} + \Pi(\mathbf{q})$$

$$\mathbf{p}^T B(\mathbf{0}) \mathbf{p} \geqslant \mu^* \| \mathbf{p} \|^2, \quad \mu^* = \text{const}$$

$$\varphi_j(\mathbf{q}, \mathbf{p}) = \left[\sum_{i=1}^{l} \frac{\partial L^*}{\partial q_{n+i}} b_{ij} + \sum_{i=1}^{l} \sum_{m=1}^{n} \left(\frac{\partial L}{\partial q_{n+i}^{\cdot}} \right)^{*} \beta_{jm}^{i} q_m^{\cdot} \right] \Bigg|_{q_j^{\cdot} = \partial H / \partial p_j}$$

Since $\varphi_j(\mathbf{q}, \mathbf{p})$ is a quadratic function of $\mathbf{p}$, it follows that

$$\varphi_j(\mathbf{q}, \mathbf{p}) = \varphi_j(\mathbf{q}, -\mathbf{p}) \tag{3.7}$$

It follows from the structure of Eqs (3.6) that replacing t by $(-t)$ in them is equivalent to the substitution $(\mathbf{q}, \mathbf{p}) \to (\mathbf{q}, -\mathbf{p})$. Denoting the vector of the right-hand sides of system (3.6) by $\mathbf{V}(\mathbf{q}, \mathbf{p})$, we have

$$\operatorname{div} \mathbf{V}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{l} \frac{\partial}{\partial q_{n+i}} \sum_{j=1}^{n} b_{ij} \frac{\partial H}{\partial p_j} + \sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial p_j}$$

whence, in view of (3.7), we get

$$\operatorname{div} \mathbf{V}(\mathbf{q}, \mathbf{p})|_{\mathbf{p} \to (-\mathbf{p})} = -\operatorname{div} \mathbf{V}(\mathbf{q}, \mathbf{p})$$

We have thus arrived at the situation considered in the proof of Theorem 1, so that we can infer the truth of Theorem 3, since equations (3.2), (3.5) are invariant to the replacement of t by $-t$.

4. Consider a non-autonomous system

$$\mathbf{x}^{\cdot\cdot} = \mathbf{f}(t, \mathbf{x}, \mathbf{x}^{\cdot})$$
$$\mathbf{f}(t, \mathbf{x}, \mathbf{x}^{\cdot}) \in C^{(0,2,2)}_{t\mathbf{x}\mathbf{x}^{\cdot}}(R \times D, D \subset R^{2n}_{\mathbf{x}\mathbf{x}^{\cdot}}) \tag{4.1}$$

Assuming that solutions of system (4.1) with initial data in a certain domain $D^*(\mathbf{x}, \mathbf{x}^{\cdot}) \subset D$ can be continued to the entire real line $t \in R$, we can generalize the concept of reversibility, defining it by the following equality:

$$\mathbf{f}(t, \mathbf{x}, \mathbf{y}) = \mathbf{f}(-t, \mathbf{x}, -\mathbf{y}) \tag{4.2}$$

We now write Eqs (4.1) in the form

$$\mathbf{x}^{\cdot} = \mathbf{y}, \quad \mathbf{y}^{\cdot} = \mathbf{f}(t, \mathbf{x}, \mathbf{y}) \tag{4.3}$$

Theorem 4. Assume that the reversible system (4.1) satisfies the following conditions:
1. There exists a bounded set of trajectories $G = \{(\mathbf{x}(t), \mathbf{x}^{\cdot}(t)) : t \in R\} \subset D$;
2. $\operatorname{mes}(G) > 0$.
Then system (4.3), as an equivalent form of system (4.1), has an integral invariant of the form

$$v^* = \int_\Omega \rho(t, \mathbf{x}, \mathbf{y})\, dx dy, \quad \Omega \subset G|_{\mathbf{x}^{\cdot} \to \mathbf{y}} = G^* \tag{4.4}$$

with density $0 < \rho(t, \mathbf{x}, \mathbf{y}) \in C^{(1,1,1)}_{t\mathbf{x}\mathbf{y}}(R \times G^*)$ bounded almost everywhere in the set $R \times G^*$.

Proof. Assuming that a solution of system (4.3) is known, we write it as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \boldsymbol{\psi}(t, t_0, \mathbf{x}_0, \mathbf{y}_0), \qquad \boldsymbol{\psi} = (\psi_1, \ldots, \psi_{2n})^T \tag{4.5}$$

Consider the auxiliary equation

$$z^{\cdot} = -z \operatorname{div}_{\mathbf{y}} \mathbf{f}(t, \mathbf{x}, \mathbf{y}), \quad \operatorname{div}_{\mathbf{y}} \mathbf{f} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_i} \tag{4.6}$$

replacing $\mathbf{x}$, $\mathbf{y}$ by the expressions obtained from (4.5). Integrating (4.6), we obtain

$$z = a\alpha(t, t_0, \mathbf{x}_0, \mathbf{y}_0), \qquad a = \text{const} \tag{4.7}$$

where, by our initial assumptions in (4.1), $\alpha \in C^{(1,1,1,1)}_{tt_0\mathbf{x}_0\mathbf{y}_0}$. Then, inverting the vector equation (4.5) and substituting the result into (4.7), we have

$$z = a\beta(t, t_0, \mathbf{x}, \mathbf{y}) \in C^{(1,1,1,1)}_{t_0 t\mathbf{x}\mathbf{y}}(R \times D) \tag{4.8}$$

By (4.6), $\beta > 0$, $\forall (t_0, t, \mathbf{x}, \mathbf{y}) \in R \times D$.
Using (4.8), we can write Eq. (4.6) in the form

$$a\left(\frac{\partial \beta}{\partial t} + \frac{\partial \beta}{\partial \mathbf{x}} \mathbf{y} + \frac{\partial \beta}{\partial \mathbf{y}} \mathbf{f}\right) = -a\beta \operatorname{div}_{\mathbf{y}} \mathbf{f} \tag{4.9}$$

Dividing both sides of (4.9) by the constant a, we get

$$\partial\beta/\partial t + \operatorname{div}(\beta \mathbf{F}) = 0, \qquad \mathbf{F} = (\mathbf{y}, \mathbf{f})^T$$

On the other hand, we have

$$\frac{d}{dt}\left(\int_\Omega \beta\, dx dy\right) = \int_\Omega \left(\frac{\partial \beta}{\partial t} + \operatorname{div}(\beta \mathbf{F})\right) dx dy$$

Hence, putting $\beta(t, 0, \mathbf{x}, \mathbf{y}) = \rho(t, \mathbf{x}, \mathbf{y})$, we conclude that there exists an integral invariant (4.4) with density $0<\rho\in C_{txy}^{(1,1,1)}$. To complete the proof, we need only observe that the fact that ρ is bounded almost everywhere in $R\times G^*$ can be proved by applying the arguments used to prove Lemma 2 to system (4.3), (4.6).

Corollaries. 1. The following estimate is true:

$$\lambda_1 \leqslant \text{mes}(\Delta_t) \leqslant \lambda_2, \quad 0<\lambda_i=\text{const}, \quad i=1,2$$

where

$$\text{mes}(\Delta_t)=\int_{\Delta_t} dxdy$$

$$0<\text{mes}(\Delta|_{t=t_0})=\text{mes}(\Delta_0)<\infty, \quad \Delta_0\subset G^*$$

2. An equilibrium position of the reversible system (4.1) cannot be asymptotically stable.
3. Under the assumptions of Theorem 3, assuming moreover that $\mathbf{f}(t,\mathbf{x},\mathbf{x}^\cdot)$ is a periodic function of t, we can conclude that Poisson stability holds almost everywhere in the set G.

The corollaries can be proved using the well-known scheme of [14, p. 214].

Remark. The proof of Theorem 4 enables one to conclude that there exists an integral invariant with density depending on t even in the most general case, when the solutions of a non-autonomous system

$$\mathbf{x}^\cdot=\mathbf{X}(t,\mathbf{x})$$

satisfy the continuation property, provided that the right-hand sides of the system are sufficiently smooth. However, the question of whether the density $\rho(t,\mathbf{x})$ is bounded in this situation remains open.

5. The Remark in Sec. 4 may be used as a guiding argument to obtain criteria for the non-existence of integral invariants of type (1.2) for autonomous systems (1.1). In particular, we have the following theorem, which is similar in content to Theorems 1 and 2 of [5]:

Theorem 5. If $\mathbf{X}(\mathbf{x})\in C^r(D(\mathbf{x}))$ $(r\geqslant 1)$ in system (1.1) and there exists a bounded (positive or negative) semi-trajectory $\gamma^+(-)=\{\mathbf{x}^*(t):t\in R^+(R^-)\}$ such that

1. $$\bar{\gamma}^{+(-)}\subset D(\mathbf{x})$$

2. $$\inf_{t\in R^+(R^-)}\int_0^t \text{div}\,\mathbf{X}(\mathbf{x}^*(\tau))\,d\tau=-\infty$$

then system (1.1) has no integral invariant of type (1.2) in the domain D.

Proof. Suppose the contrary. Then there exists a positive function $\rho(\mathbf{x})\in C^1(D(\mathbf{x}))$ which satisfies Liouville's equation (1.3). Expressing this positive function as $\rho^\cdot=-\rho\,\text{div}\mathbf{X}$ and integrating it for $\mathbf{x}=\mathbf{x}^*(t)$, we obtain

$$\rho(\mathbf{x}^*(t))=\rho_0\exp\left(-\int_0^t \text{div}\,\mathbf{X}(\mathbf{x}^*(\tau))\,d\tau\right)$$
$$\rho|_{t=0}=\rho_0>0 \tag{5.1}$$

By condition 2 of the theorem, Eq (5.1) implies that

$$\sup_{t\in R^+(R^-)}\rho(\gamma^{+(-)}(\mathbf{x}^*))=\infty \tag{5.2}$$

On the other hand, since $\bar{\gamma}^{+(-)}$, as the closure of the bounded semi-trajectory $\gamma^{+(-)}$, is compact, and moreover $\bar{\gamma}^{+(-)}\subset D$, $\rho(\mathbf{x})\in C^1(D)$, it follows that $\max\rho(\bar{\gamma}^{+(-)})<\infty$, contradicting (5.2). This completes the proof of Theorem 5.

Corollary. (see [5]). Let $\mathbf{x}=0$ be an equilibrium position of system (1.1) near which the system can be written in the form

$$\mathbf{x}^\cdot=A\mathbf{x}+o(\|\mathbf{x}\|), \quad A=(a_{ij}), \quad i,j=1,2,\ldots,n$$

Then, if $\operatorname{tr} A \neq 0$, system (1.1) has no integral invariant of type (1.2) in the neighbourhood of $\mathbf{x} = 0$.

Theorem 5 should be applied when a bounded particular solution of (1.1), not necessarily reducible to an equilibrium position, is known.

In conclusion, we note that in the context of Poisson stability the reversibility of a system (invariance under the substitution of $-t$ for t) may be used as a certain equivalent of invariant measure.

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